

# VITUSHKIN-TYPE THEOREMS

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**ABSTRACT.** It is shown that for a subset  $A \subset \mathbb{R}^n$  that has the global Gabrielov property, a Vitushkin-type estimate holds. Concrete examples are given for sub-level sets of certain classes of functions.

## 1. INTRODUCTION

The metric entropy of a subset  $A \subset \mathbb{R}^n$  can be bounded in terms of the  $i$ -dimensional “size” of  $A$ . Indeed, the theory of multi-dimensional variations, developed by Vitushkin [7, 8], Ivanov [3], and other, provides a bound by measuring the  $i$ -dimensional “size” of  $A$  in terms of its variations.

Let us recall a general definition of the metric entropy of a set. Let  $X$  be a metric space,  $A \subset X$  a relatively compact subset. For every  $\varepsilon > 0$ , denote by  $M(\varepsilon, A)$  the minimal number of closed balls of radius  $\varepsilon$  in  $X$ , covering  $A$  (note that this number does exist because  $A$  is relatively compact). The real number  $H_\varepsilon(A) = \log M(\varepsilon, A)$  is called the  $\varepsilon$ -entropy of the set  $A$ . In our setting, we assume  $X = \mathbb{R}^n$ , and it will be convenient to modify slightly this definition, and consider coverings by the  $\varepsilon$ -cubes  $Q_\varepsilon$ , which are translations of the standard  $\varepsilon$ -cube,  $Q_\varepsilon^n = [0, \varepsilon]^n$ , that is, the  $\frac{\varepsilon}{2}$ -ball in the  $\ell_\infty$  norm.

The following inequality, which we refer to as Vitushkin’s bound, bounds the metric entropy of a set  $A$  in terms of its multi-dimensional variations, that is, for every  $A \subset \mathbb{R}^n$  it holds

$$M(\varepsilon, A) \leq c(n) \sum_{i=0}^n V_i(A)/\varepsilon^i, \quad (1.1)$$

where the  $i$ -th variation of  $A$ ,  $V_i(A)$ , is the average of the number of connected components of the section  $A \cap P$  over all  $(n - i)$ -affine planes  $P$  in  $\mathbb{R}^n$ . In particular, our definition of the metric entropy implies that the last term in (1.1) has the form  $\mu_n(A)/\varepsilon^n$ , where  $\mu_n(A)$  denotes the  $n$ -dimensional Lebesgue measure (or the volume) of the set  $A$ .

Vitushkin’s bound is sometimes considered as a difficult result, mainly because of the so-called multi-dimensional variations which are used. However, in some cases (cf. [10, 9, 2]) the

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proof is indeed very short and transparent. In this note, we present a Vitushkin-type theorem, which works in situations where we can control the number of connected components of the sections  $A \cap P$  over all  $(n-i)$ -affine planes  $P$ . In section 2 we present our main observation that we can replace the  $i$ -th variation  $V_i(A)$  of  $A$ , with an upper bound on the number of connected components of the section  $A \cap P$  over all  $(n-i)$ -affine planes  $P$ . As in some cases, it is easier to compute this upper bound rather than  $V_i$  which is the average. In Section 3, we extend Vitushkin's bound for semi-algebraic sets, to sub-level sets of functions, for which we have a certain replacement of the polynomial Bézout theorem. These results can be proved using a general result of Vitushkin in [7, 8] through the use of multi-dimensional variations. However, in our specific case the results below are much simpler and shorter and produce explicit ("in one step") constants.

## 2. GABRIELOV PROPERTY AND VITUSHKIN'S BOUND

In this section we establish a relation between Vitushkin's bound and the Gabrielov property of a set  $A$ . We show that an a priori knowledge about the maximal number of connected components of a set  $A$ , intersected with every  $\ell$ -affine plane in  $\mathbb{R}^n$ , allows us to estimate the metric entropy of  $A$ .

More precisely, we say that a subset  $A \subset \mathbb{R}^n$  has the local Gabrielov property if for  $a \in A$  there exist a neighborhood  $U$  of  $a$  and an integer  $\hat{C}_\ell$  such that for every  $\ell$ -affine plane  $P$ , the number of connected components of  $U \cap A \cap P$  is bounded by  $\hat{C}_\ell$ . If we can take  $U = \mathbb{R}^n$ , we say that  $A$  has the global Gabrielov property. For example, every tame set has the local Gabrielov property (for more details see [10]).

The following theorem is applicable to arbitrary subsets  $A \subset Q_1^n$ . The boundary  $\partial A$  of  $A$  is defined as the intersection of the closures of  $A$  and of  $Q_1^n \setminus A$ .

**Theorem 1** (Vitushkin-type theorem). *Let  $A \subset Q_1^n$  and let  $0 < \varepsilon \leq 1$ . Assume that the boundary  $\partial A$  of  $A$  has the global Gabrielov property, with explicit bound  $\hat{C}_\ell$  for  $0 \leq \ell \leq n$ . Then*

$$M(\varepsilon, A) \leq C_0 + C_1/\varepsilon + \cdots + C_{n-1}/\varepsilon^{n-1} + \mu_n(A)/\varepsilon^n,$$

where  $C_t := \hat{C}_{n-t} 2^t \binom{n}{t}$ .

*Proof.* Let us subdivide  $Q_1^n$  into adjacent  $\varepsilon$ -cubes  $Q_\varepsilon^n$ , with respect to the standard Cartesian coordinate system. Each  $Q_\varepsilon^n$ , having a non-empty intersection with  $A$ , is either entirely contained in  $A$ , or it intersects the boundary  $\partial A$  of  $A$ . Certainly, the number of those cubes  $Q_\varepsilon^n$ , which are entirely contained in  $A$ , is bounded by  $\mu_n(A)/\mu_n(Q_\varepsilon) = \mu_n(A)/\varepsilon^n$ . In the other case, in which  $Q_\varepsilon^n$  intersects  $\partial A$ , it means that there exist faces of  $Q_\varepsilon^n$  that have a non-empty intersection with  $\partial A$ . Among all these faces, let us take one with the smallest

dimension  $s$ , and denote it by  $F$ . In other words, there exists an  $s$ -face  $F$  of the smallest dimension  $s$  that intersects  $\partial A$ , for some  $s = 0, 1, \dots, n$ . Let us fix an  $s$ -affine plane  $V$ , which corresponds to  $F$ . Then, by the minimality of  $s$ ,  $F$  contains completely some of the connected components of  $\partial A \cap V$ , otherwise  $\partial A$  would intersect a face of  $Q_\varepsilon^n$  of a dimension strictly less than  $s$ . By our assumption, the number of connected components with respect to an  $s$ -affine plane is bounded by  $\hat{C}_s$ . According to the subdivision of  $Q_1^n$  to  $Q_\varepsilon$  cubes, we have at most  $(\frac{1}{\varepsilon} + 1)^{n-s}$   $s$ -affine planes with respect to the same  $s$  coordinates, and the number of different choices of  $s$  coordinates is  $\binom{n}{s}$ . It means that the number of cubes, that have an  $s$ -face  $F$  which contains completely some connected component of  $A \cap V$ , is at most

$$\hat{C}_s \binom{n}{s} \left(\frac{1}{\varepsilon} + 1\right)^{n-s} \leq \hat{C}_s 2^{n-s} \binom{n}{s} / \varepsilon^{n-s}.$$

Let us define the constant

$$C_{n-s} := \hat{C}_s 2^{n-s} \binom{n}{s}.$$

Note that  $C_0$  is the bound on the number of cubes that contain completely some of the connected components of  $A$ . Thus, we have

$$M(\varepsilon, A) \leq C_0 + C_1/\varepsilon + \dots + C_{n-1}/\varepsilon^{n-1} + \mu_n(A)/\varepsilon^n.$$

This completes the proof. □

### 3. ENTROPY ESTIMATES OF SUB-LEVEL SETS

In this section we extend Vitushkin's bound to sub-level sets of certain natural classes of functions, beyond polynomials. We do so by “counting” the singularities of these functions, and then bounding the number of the connected components of their sub-level set through the number of singularities.

We start with a general simple “meta”-lemma, which implies, together with a specific computation of the bound on the number of singularities, all our specific results below. Consider a class of functions  $\mathcal{F}$  on  $\mathbb{R}^n$ . We assume that  $\mathcal{F}$  is closed with respect to taking partial derivatives, restrictions to affine subspaces of  $\mathbb{R}^n$ , and with respect to sufficiently rich perturbations. There are many classes of functions that comply with this condition, for example, we may speak about the class of real polynomials of  $n$  variables and degree  $d$ , and the classes considered below in this section. Assume that for each  $f_1, \dots, f_n \in \mathcal{F}$  the number of non-degenerate solutions of the system

$$f_1 = f_2 = \dots = f_n = 0,$$

is bounded by the constant  $C(D(f_1, \dots, f_n))$ , where  $D(f_1, \dots, f_n)$  is a collection of “combinatorial” data of  $f_i$ , like degrees, which we call a “Diagram” of  $f_1, \dots, f_n$ . We assume that

the diagram is stable with respect to the deformations we use. In each of the examples below we define the appropriate diagram specifically.

Let  $f \in \mathcal{F}$ . Denote by

$$W = W(f, \rho) = \{x \in Q_1^n : f(x) \leq \rho\}, \quad (3.1)$$

the  $\rho$ -sub-level set of  $f$ , and let  $\hat{C}_s = C(D(\frac{\partial f(x)}{\partial x_1}, \dots, \frac{\partial f(x)}{\partial x_s}))$ ,  $s = 1, \dots, n$ .

**Lemma 2.** *The boundary  $\partial W$  has the global Gabrielov property, i.e. the number of connected components of  $\partial W \cap P$ , where  $P$  is an  $s$ -affine plane in  $\mathbb{R}^n$ , is bounded by  $\hat{C}_s$ .*

*Proof.* We may assume that  $P$  is a parallel translation of the coordinate plane in  $\mathbb{R}^n$  generated by  $x_{j_1}, \dots, x_{j_s}$ . Inside each connected component of  $\partial W \cap P$  there is a critical point of  $f$  restricted to  $P$  (its local maximum or minimum), which is defined by the system of equations

$$\frac{\partial f(x)}{\partial x_{j_1}} = \dots = \frac{\partial f(x)}{\partial x_{j_s}} = 0.$$

After a small perturbation of  $f$ , we can always assume that all such critical points are non-degenerate. Hence by the assumptions above, the number of these points, and therefore the number of connected components, is bounded by  $\hat{C}_s$ .  $\square$

In other words,  $W$  has the Gabrielov property, with explicit bound  $\hat{C}_\ell$ . Therefore, Theorem 1 can be applied to this set, and under the assumptions above, we have

**Corollary 3.** *Let  $0 < \varepsilon \leq 1$ . Then*

$$M(\varepsilon, W) \leq C_0 + C_1/\varepsilon + \dots + C_{n-1}/\varepsilon^{n-1} + \mu_n(W)/\varepsilon^n,$$

where  $C_t := \hat{C}_{n-t} 2^t \binom{n}{t}$ .

#### 4. CONCRETE BOUNDS ON $\hat{C}_s$

In view of Corollary 3, our main goal now is to give concrete bounds on constants  $\hat{C}_s$  in specific situations.

**4.1. Polynomials and Bézout's theorem.** Let  $p(x) = p(x_1, \dots, x_n)$  be a polynomial in  $\mathbb{R}^n$  of degree  $d$ . We consider the sub-level set  $W(p, \rho)$  as defined in (3.1). Clearly, in this situation, by Bézout's theorem, we have

$$\hat{C}_s(W(p, \rho)) \leq (d - s)^s.$$

**4.2. Laurent polynomials and Newton polytypes.** Let  $\alpha \in \mathbb{Z}^n$ . A Laurent monomial in the variables  $x_1, \dots, x_n$  is  $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ . A Laurent polynomial is a finite sum of Laurent monomials,

$$p(x) = p(x_1, \dots, x_n) = \sum_{\alpha \in A \subset \mathbb{Z}^n} a_\alpha x^\alpha.$$

The Newton polytope of  $p$  is the polytope

$$N(p) = \text{conv}\{\alpha \in \mathbb{Z}^n \mid a_\alpha \neq 0\}.$$

A natural generalization of the Bézout bound above is the following Bernstein-Kušnirenko bound for polynomial systems with the prescribed Newton polytope.

**Theorem 4** ([5, 1]). *Let  $f_1, \dots, f_n$  be Laurent polynomials with the Newton polytope  $N \subset \mathbb{R}^n$ . Then the number of non-degenerate solutions of the system*

$$f_1 = f_2 = \cdots = f_n = 0,$$

*is at most  $n! \text{Vol}_n(N)$ .*

The Newton polytope of a general polynomial of degree  $d$  is the simplex

$$\Delta_d = \{\alpha \in \mathbb{R}^n, |\alpha| \leq d\}.$$

Its volume is  $d^n/n!$ , and the bound of Theorem 4 coincides with the Bézout's bound  $d^n$ . The notion of the Newton polytope is connected to a representation of the polynomial  $p$  in a fixed coordinate system  $x_1, \dots, x_n$  in  $\mathbb{R}^n$ . As we perform a coordinate changes (which may be necessary when we restrict a polynomial  $p$  to a certain affine subspace  $P$  of  $\mathbb{R}^n$ ),  $N(p)$  may change strongly. However, in Theorem 1 we restrict our functions only to affine subspaces  $P$  spanned by a part of the standard basis vectors in a fixed coordinate system. The following lemma describes the behavior of the Newton polytope of  $p$  under such restrictions, and under partial differentiation.

**Lemma 5.** *The Newton polytope  $N(\frac{\partial p}{\partial x_i})$  is  $N_i(p)$  obtained by a translation of  $N(p)$  to the vector  $-e_i$ , where  $e_1, \dots, e_n$  are the vectors of the standard basis in  $\mathbb{R}^d$ . The Newton polytope  $N(p|P)$  of a restriction of  $p$  to  $P$ , where  $P$  is a translation of a certain coordinate subspace, is contained in the projection  $\pi_P(N(p))$  of  $N(p)$  on  $P$ .*

*Proof.* The proof of the first claim is immediate. In a restriction of  $p$  to  $P$ , we substitute some of the  $x_i$ 's for their specific values. The degrees of the free variables remain the same.  $\square$

For a Newton polytope  $N \subset \mathbb{R}^n$  define

$$C_s(N) := \max\{\text{Vol}_s(N_{P_s}) \mid s\text{-dimensional coordinate subspaces } P_s\},$$

where  $N_{P_s}$  is the convex hull of the sets  $\pi_{P_s}(N_i(p))$ , for all the coordinate directions in  $P_s$ . Here,  $\pi_{P_s}(N_i(p))$  is the projection of  $N$  to  $P_s$ , shifted by  $-1$  in one of coordinate directions  $x_i$  in  $P_s$ .

**Theorem 6.** *Let  $p$  be a Laurent polynomial with the Newton polytope  $N$ . Then for  $s = 1, \dots, n$*

$$\hat{C}_s(W(p, \rho)) \leq \frac{C_s(N)}{s!}.$$

*Proof.* According to the *proof* of Theorem 1, the constants  $\hat{C}_s(W(p, \rho))$  do not exceed the number of solutions in  $Q_1^n$  of the system  $\frac{\partial p(x)}{\partial x_{j_1}} = \dots = \frac{\partial p(x)}{\partial x_{j_s}} = 0$ . Now application of Theorem 4, Lemma 5, and of the definition of  $C_s(N)$  above, completes the proof.  $\square$

An important example is provided by “multi-degree  $d$ ” polynomials. A polynomial  $p(x) = p(x_1, \dots, x_n)$  is called multi-degree  $d$ , if each of its variable enters  $p$  with degrees at most  $d$ . The total degree of such  $p$  may be  $nd$ . In particular, multi-linear polynomials contain each variable with the degree at most one. Multi-linear polynomials appear in various problems in Mathematics and Computer Science, in particular, since the determinant of a matrix is a multi-linear polynomial in its entries.

**Theorem 7.** *Let  $p$  be a multi-degree  $d$  polynomial. Then for  $s = 1, \dots, n$*

$$\hat{C}_s(W(p, \rho)) \leq \frac{d^s}{s!}.$$

*Proof.* The Newton polytope  $N(p)$  of a multi-degree  $d$  polynomial  $p$  is a cube  $Q_d^n$  with the edge  $d$ . Its projection to each  $P_s$  is  $Q_d^s$ . After the shift by  $-1$  in one of the coordinate directions in  $P_s$ , it remains in  $Q_d^s$ . So  $Q_d^s$  can be taken as the common Newton polytope of  $\frac{\partial p(x)}{\partial x_{j_1}}, \dots, \frac{\partial p(x)}{\partial x_{j_s}}$ . Application of Theorem 6 completes the proof.  $\square$

**4.3. Quasi-polynomials and Khovanskii’s theorem.** Let  $f_1, \dots, f_k \in (\mathbb{C}^n)^*$  be a pairwise different set of complex linear functionals  $f_j$  which we identify with the scalar products  $f_j \cdot z$ ,  $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ . We shall write  $f_j = a_j + ib_j$ . A quasi-polynomial is a finite sum

$$p(z) = \sum_{j=1}^k p_j(z) e^{f_j \cdot z},$$

where  $p_j \in \mathbb{C}[z_1, \dots, z_n]$  are polynomials in  $z$  of degrees  $d_j$ . The degree of  $p$  is  $m = \deg p = \sum_{j=1}^k (d_j + 1)$ .

Below we consider  $p(x)$  for the real variables  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ , and we are interested in the following sub-level set of  $p$  which is defined as  $\{x \in Q_1^n : |p(x)| \leq \rho\}$ . Denote  $q(x) = |p(x)|^2$ , then this sub-level set is also defined by  $W(q, \rho^2)$ .

A simple observation that  $q(x) = |p(x)|^2 = p(x)\bar{p}(x)$  tells us that we can rewrite  $q$  as follows

**Lemma 8.**  $q(x)$  is a real exponential trigonometric quasi-polynomial with  $P_{ij}, Q_{ij}$  real polynomials in  $x$  of degree  $d_i + d_j$ , and at most  $\kappa := k(k+1)/2$  exponents, sinus and cosinus elements. Moreover,

$$q(x) = \sum_{0 \leq i \leq j \leq k} e^{\langle a_i + a_j, x \rangle} [P_{ij}(x) \sin \langle b_{ij}, x \rangle + Q_{ij}(x) \cos \langle b_{ij}, x \rangle],$$

where  $b_{ij} = b_i - b_j \in \mathbb{R}^n$ .

Now, we need to bound the singularities of  $q$ . This can be done using the following theorem due to Khovanskii, which gives an estimate of the number of solutions of a system of real exponential trigonometric quasi-polynomials.

**Theorem 9** (Khovanskii bound [4], Section 1.4). *Let  $P_1 = \dots = P_n = 0$  be a system of  $n$  equations with  $n$  real unknowns  $x = x_1, \dots, x_n$ , where  $P_i$  is polynomial of degree  $m_i$  in  $n+k+2p$  real variables  $x, y_1, \dots, y_k, u_1, \dots, u_p, v_1, \dots, v_p$ , where  $y_j = \exp \langle a_j, x \rangle$ ,  $j = 1, \dots, k$  and  $u_q = \sin \langle b_q, x \rangle$ ,  $v_q = \cos \langle b_q, x \rangle$ ,  $q = 1, \dots, p$ . Then the number of non-degenerate solutions of this system in the region bounded by the inequalities  $|\langle b_q, x \rangle| < \pi/2$ ,  $q = 1, \dots, p$ , is finite and less than*

$$m_1 \cdots m_n \left( \sum m_i + p + 1 \right)^{p+k} 2^{p+(p+k)(p+k-1)/2}.$$

Clearly, all the partial derivatives  $\frac{\partial q(x)}{\partial x_j}$  have exactly the same form as  $q$ . Therefore, Khovanskii's theorem gives the following bound on the number of critical points of  $q$ . More precisely, we have

**Lemma 10.** *Let  $V$  be a parallel translation of the coordinate subspace in  $\mathbb{R}^n$  generated by  $x_{j_1}, \dots, x_{j_s}$ . Then the number of non-degenerate real solutions in  $V \cap Q_\rho^n$  of the system*

$$\frac{\partial q(x)}{\partial x_{j_1}} = \dots = \frac{\partial q(x)}{\partial x_{j_s}} = 0,$$

is at most

$$\left( \frac{2}{\pi} \sqrt{s} \rho \lambda \right)^s \prod_{r=1}^s (d_{j_r} + d_{i_r}) \left( \sum_{r=1}^s d_{j_r} + d_{i_r} + 2\kappa + 1 \right)^{2\kappa} 2^{\kappa + (2\kappa)(2\kappa-1)/2},$$

where  $\lambda := \max \|b_{ij}\|$  is the maximal frequency in  $q$ .

*Proof.* The following geometric construction is required by the Khovanskii bound: Let  $Q_{ij} = \{x \in \mathbb{R}^n, |\langle b_{ij}, x \rangle| \leq \frac{\pi}{2}\}$  and let  $Q = \bigcap_{0 \leq i \leq j \leq k} Q_{ij}$ . For every  $B \subset \mathbb{R}^n$  we define  $M(B)$  as the minimal number of translations of  $Q$  covering  $B$ . For an affine subspace  $V$  of  $\mathbb{R}^n$  we define  $M(B \cap V)$  as the minimal number of translations of  $Q \cap V$  covering  $B \cap V$ . Notice that for  $B = Q_r^n$ , a cube of size  $r$ , we have  $M(Q_r^n) \leq (\frac{2}{\pi} \sqrt{n} r \lambda)^n$ . Indeed,  $Q$  always contains a ball of

radius  $\frac{\pi}{2\lambda}$ . Now, applying the Khovanskii's theorem on the system

$$\frac{\partial q(x)}{\partial x_{j_1}} = \dots = \frac{\partial q(x)}{\partial x_{j_s}} = 0,$$

we get that the number of non-degenerate real solutions in  $V \cap Q_\rho^n$  is at most

$$\left(\frac{2}{\pi}\sqrt{s\rho\lambda}\right)^s \prod_{r=1}^s (d_{j_r} + d_{i_r}) \left(\sum_{r=1}^s d_{j_r} + d_{i_r} + 2\kappa + 1\right)^{2\kappa} 2^{\kappa+(2\kappa)(2\kappa-1)/2}.$$

Note that this bound is given in term of the “diagram” of  $q$ , and therefore of  $p$ .  $\square$

**Theorem 11.** *Let  $p(x)$  a real quasi-polynomial (as described above). Then for  $s = 1, \dots, n$*

$$\begin{aligned} & \hat{C}_s(W(q, \rho^2)) \\ & \leq \left(\frac{2}{\pi}\sqrt{s\rho\lambda}\right)^s \prod_{r=1}^s (d_{j_r} + d_{i_r}) \left(\sum_{r=1}^s d_{j_r} + d_{i_r} + 2\kappa + 1\right)^{2\kappa} 2^{\kappa+(2\kappa)(2\kappa-1)/2}, \end{aligned}$$

where  $q(x) = |p(x)|^2$ .

**4.4. Exponential polynomials and Nazarov's lemma.** In a particular case where  $p$  is an exponential polynomial, that is,

$$p(t) = \sum_{k=0}^m c_k e^{\lambda_k t},$$

where  $c_k, \lambda_k \in \mathbb{C}$ ,  $t \in \mathbb{R}$ . We can avoid Khovanskii's theorem and instead use the following result of Nazarov [6, Lemma 4.2], which gives a bound on the local distribution of zeroes of an exponential polynomial.

**Lemma 12.** *The number of zeroes of  $p(z)$  inside each disk of radius  $r > 0$  does not exceed*

$$4m + 7\hat{\lambda}r,$$

where  $\hat{\lambda} = \max |\lambda_k|$ .

Note that this result is applicable only in dimension 1. Let  $B \subset \mathbb{R}$  be an interval. Therefore, the number of real solutions of  $p(t) \leq \rho$  inside the interval  $B$  does not exceed Nazarov's bound, that is,  $4m + 7\hat{\lambda}\mu_1(B)$ , which gives us

$$\hat{C}_1(W(p, \rho)) \leq 4m + 7\hat{\lambda}.$$

For the case of a real exponential polynomial  $p(t) = \sum_{k=0}^m c_k e^{\lambda_k t}$ ,  $c_k, \lambda_k \in \mathbb{R}$ , we get an especially simple and sharp result, as the number of zeroes of a real exponential polynomial is always bounded by its degree  $m$  (indeed, the “monomials”  $e^{\lambda_k t}$  form a Chebyshev system on each real interval).



**4.5. Semialgebraic and tame sets.** We conclude with a remark about even more general settings for which Theorem 1 is applicable.

A set  $A \subset \mathbb{R}^n$  is called semialgebraic, if it is defined by a finite sequence of polynomial equations and inequalities, or any finite union of such sets. More precisely,  $A$  can be represented in a form  $A = \bigcup_{i=1}^k A_i$  with  $A_i = \bigcap_{j=1}^{j_i} A_{ij}$ , where each  $A_{ij}$  has the form

$$\{x \in \mathbb{R}^n : p_{ij}(x) > 0\} \text{ or } \{x \in \mathbb{R}^n : p_{ij}(x) \geq 0\},$$

where  $p_{ij}$  is a polynomial of degree  $d_{ij}$ . The diagram  $D(A)$  of  $A$  is the collective data

$$D(A) = (n, k, j_1, \dots, j_k, (d_{ij})_{i=1, \dots, k, j=1, \dots, j_i}).$$

A classical result tells us that the number of connected components of a plane section  $A \cap P$  is uniformly bounded. More precisely, we have

**Theorem 13** ([10]). *Let  $A \subset \mathbb{R}^n$  be a semialgebraic set with diagram  $D(A)$ . Then the number of connected components of  $A \cap P$ , where  $P$  is an  $\ell$ -affine plane of  $\mathbb{R}^n$ , is bounded by*

$$\hat{C}_\ell \leq \frac{1}{2} \sum_{i=1}^k (d_i + 2)(d_i + 1)^{\ell-1},$$

where  $d_i = \sum_{j=1}^{j_i} d_{ij}$ .

In other words, Theorem 13 says that any semialgebraic set has the Gabrielov property.

However, not only semialgebraic sets, but a very large class of sets has the Gabrielov property. These sets are called *tame sets*. The precise definition of these sets and the fact that they satisfy the Gabrielov property can be found, in particular, in [10].

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